JOURNAL OF COMPUTATIONAL PHYSICS, 12 553–556 (1973)

Note

A New Form of Artificial Viscosity: Postscript*

In a previous paper [1], we introduced a new form of artificial vicosity (q), which performs better in a one-dimensional Lagrangian hydrodynamics code by substituting the Rankine–Hugoniot (R–H) relations directly into the form originally suggested by Von Neumann and Richtmyer [2]. The new form was given by the following:

$$q_v = \rho c^2 \, \Delta U \sqrt{-\Delta p \, \Delta(1/\rho)}. \tag{1}$$

However, this new form is difficult to program in multidimensional or Eulerian programs, because of the presence of interfaces between different materials. Across interfaces, density (or energy) gradients cannot be used in the R–H relations. This note will suggest forms that will not have such difficulties.

From the first R–H equation, we have

$$\rho_0 \dot{\xi} = \sqrt{-\Delta p \left(\Delta \frac{1}{\rho}\right)^{-1}}, \tag{2}$$

where $\dot{\xi}$ is the shock speed, ρ_0 is the unshocked material density, and Δp and $\Delta(1/\rho)$ are the changes in pressure and inverse density across the shock. From the second R–H equation we have

$$\Delta U = \sqrt{-\Delta p \, \Delta(1/\rho)}, \tag{3}$$

where ΔU is the change in material velocity across the shock. Combining (1) and (2) yields

$$\Delta U = \frac{\Delta p}{\rho_0 \dot{\xi}}. \tag{4}$$

Since pressure is a continuous variable across interfaces, this would be a convenient relation to program into the formulation for q. Unfortunately, $\dot{\xi}$ is usually an unknown. Instead, let us examine the replacement of ΔU in a q as follows:

$$\Delta U \Rightarrow \frac{\Delta p}{\rho c_s}. \tag{5}$$

* This work was performed under the auspices of the U.S. Atomic Energy Commission.

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For physical systems involving potentials, such as gravitational potentials in astrophysical systems, it might be necessary to use only that portion of Δp which can give rise to velocity changes (i.e., that portion not balanced by a potential gradient). The sun is an example of a system possessing a large pressure gradient in balance with a gravitational potential—yet, there is no velocity change.

For infinitely weak shocks (sound waves), $\Delta U = \Delta p/(\rho c_s)$. However, for strong shocks, ρc_s will be smaller than $\rho_0 \dot{\xi}$ ahead of the shock, but the opposite will be true behind the shock. Averaged over a shock of finite width, one might anticipate that using (5) in a q formulation would not change the overall behavior too much. However, it would produce increased damping at the leading edge of a shock, and less at the trailing edge. Numerical experiments indicate that a good form of q is

$$q_E = \rho c^2 \mid \Delta U \mid^{3/2} \left| \frac{\Delta p}{\rho c_s} \right|^{1/2}. \tag{6}$$

A larger power in the exponent of $(\Delta p/\rho c_s)$ results in a noisy solution, consistent with the fact that (5) results in less damping at the shock's trailing edge. The q in (6) is sensitive to gradients in zone quantities (pressure) and grid quantities (velocities). Calculations using (6) give more accurate results than the standard q ($q = \rho c^2 \mid \Delta U \mid^2$) at interfaces and reflecting walls, and anomalous heating is essentially eliminated for an adiabatically compressed sphere (which has no pressure gradients).

K. Trigger [3] has suggested that time derivatives might also be programmed conveniently in various computer codes. However, one wants this time derivative to be related to the spatial derivative, so that the shock will still be spread over a certain number of zones. Thus, one might take advantage of the characteristic solution nature of shock wave propagation. Then, for plane wave shocks we can use

$$\frac{\Delta f}{\Delta x} = \frac{1}{\dot{\xi}} \frac{\Delta f}{\Delta t}, \tag{7}$$

since $f = f(x - \dot{\xi}t)$, where f is velocity, pressure, density, etc. Again, it might be necessary to use such a substitution to a low order, if $\dot{\xi}$ is approximated by c_s.

The form for q_E is generally appropriate for Eulerian and Lagrangian cydes. A more accurate form for Lagrangian codes is possible by using the time derivative just mentioned in (7). Let

$$\frac{\Delta(1/\rho)}{\Delta x} \Rightarrow \frac{1}{c_s} \frac{\Delta(1/\rho)}{\Delta t}. \tag{8}$$

Then we obtain by modifying q_p:

$$q_L = \rho c^2 \mid \Delta U \mid \sqrt{\left| \Delta p \frac{\Delta x}{c_s} \frac{\Delta(1/\rho)}{\Delta t} \right|}.$$

As was the case with q_E, $\dot{\xi}^{1/2}$ has been approximated by $c_s^{1/2}$. However, the approximation of ρ_0 by ρ has not been necessary. Because of this (and because damping is more sensitive to zonal quantities), one might anticipate that q_L would produce more accurate results than q_E, for Lagrangian calculations.

TABLE I

Shock Reflection at a Rigid Wall

$$q_E = 2.0\rho \mid \Delta U \mid^{3/2} \left| \frac{\Delta p}{\rho c_s} \right|^{1/2} + 0.2\rho c_s \mid \Delta U \mid^{3/4} \left| \frac{\Delta p}{\rho c_s} \right|^{1/4}$$

Zone	Pressure (Mbar)	Energy (Mbar cc/cc)
1[a]	7.93	1.10
2	7.94	0.95
3	7.92	0.87
4	7.94	0.88
5	7.95	0.91
6	7.93	0.93
7	7.96	0.94
8	7.94	0.95
9	7.96	0.96
10	7.96	0.96
11	7.95	0.95

[a] Zone 1 is next to the reflecting wall.

In Table I we present the q_E results of a $\gamma = 1.4$ gas ($\rho_0 = 0.008$ g/cc, $p_0 = 1.6 \times 10^{-5}$ Mbar), after an infinite pressure (1 Mbar) has been applied to one end of a one-dimensional slab and bounced off a perfectly reflecting wall. The analytic solution for the reflected pressure is 8.0 Mbar, and for the reflected energy, 20/21 (approximately = 0.95) Mbar–cc/cc. Thus, the worst error in the energy is approximately 15%. The same problem, run with the standard q (using the same constants to multiply the quadratic and linear components), resulted in an error of 60%. The improved q reported earlier [1],

$$(q_v = pc^2 \, \Delta U \sqrt{-\Delta p \, \Delta(1/\rho)}),$$

resulted in an error of 10%. Thus, the thermodynamics are almost as good as that obtained with q_v, and the programming is much simpler. For the one-dimensional

Lagrangian code used here, the programming is only important for one-zone regions. But for other codes (Eulerian and multidimensional) the significance of simpler programming may be quite important. There are many kinds of such computer programs, and I have not attempted any investigations using Eq. (5) with any of them. A wide variety of tests were run with a one-dimensional Lagrangian code, in addition to the one presented here in the example; in every case the results were like those of this example. That is, the new q of Eq. (6) was almost as good as q_p, and much better than q_u. Similarly the results for q_L were better than the results for q_E, but not as good as those for q_p.

Current experiments indicate that q_L may be attractive for elastic calculations, even though pressure is not always continuous across material interfaces. For such calculations it has been advantageous to replace

$$\Delta x \left(\frac{dU}{dx}\right) \quad \text{by} \quad \Delta x(\nabla \cdot U).$$

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RECEIVED: November 16, 1972

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